

RAPID COMMUNICATION

On 4-dimensional Lorentz-structures, Dark energy and Exotic smoothness

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Abstract. Usually, the topology of a 4-manifolds M is restricted to admit a global hyperbolic structure $\Sigma \times \mathbb{R}$. The result was obtained by using two conditions: existence of a Lorentz structure and causality (no time-like closed curves). In this paper we study the influence of the smoothness structure to show its independence of the two conditions. Then we obtain the possibility for a topology-change of the 3-manifold Σ keeping fix its homology. We will study the example $S^3 \times \mathbb{R}$ with an exotic differential structure more carefully to show some implications for cosmology. Especially we obtain an interpretation of the transition in topology as dark energy.

PACS numbers: 04.20.Gz, 98.80.Jk, 95.36.+x

Submitted to: *J. Phys. A: Math. Gen.*

A manifold admits a Lorentzian structure if (and only if) there is a line element field (i.e. a non-vanishing vector field of arbitrary sign, see Hawking and Ellis [1]). This condition is equivalent to the existence of a codimension-1 foliation [2]. Then a compact manifold admits a Lorentzian metric if the Euler characteristics vanishes. In case of a compact 4-manifold one obtains a multiple-connected manifold. But then one has the problem of causality (time loops etc., see also the topological censorship by Schleich and Witt [3]). Usually causality can be implemented on a non-compact 4-manifold. Furthermore, detailed measurements of the background radiation by the satellites COBE and WMAP [4] enforces us to assume that the space-like component is a compact 3-manifold Σ . Both conditions, Lorentz structure and causality, can be trivially fulfilled on 4-manifolds $\Sigma \times \mathbb{R}$. But, is this choice unique?

1. Lorentz structure and smoothness

Non-compactness of the 4-manifold $\Sigma \times \mathbb{R}$ as the result of the causality condition is a purely topological condition. The second condition, the existence of a Lorentz structure, is related to the structure of the tangent bundle $T(\Sigma \times \mathbb{R})$, i.e. to the smoothness structure.

The maximal differentiable atlas of a manifold is called its smoothness structure. It is unique up to diffeomorphisms. In dimensions smaller than 4 there is a unique smoothness structure [5, 6] whereas in dimensions greater than 4 we have finitely many different (=non-diffeomorphic) smoothness structures [7]. Only in dimension 4 there is the possibility for infinite many different smoothness structures with tremendous implication for quantum field theory. For a deeper insight we refer to the book [8].

If two manifolds are homeomorphic but non-diffeomorphic, they are **exotic** to each other. The smoothness structure is called an **exotic smoothness structure**. Among the smoothness structures there is one distinguish element, the **standard structure**. This structure is given by a smooth embedding of the 4-manifold into some Euclidean space \mathbb{R}^N for $N > 7$. We remark further that different smoothness structures have to represent different physical situations leading to different measurable results. But it should be stressed that *exotic smoothness is not exotic physics*. Exotic smoothness is a mathematical structure which should be further explored to understand its physical relevance.

In case of the Lorentz structure, we have to deal with the tangent bundle $T(\Sigma \times \mathbb{R})$. We mentioned above that the existence of a Lorentz structure is connected with the existence of a line element. This line element exists only if the tangent bundle admits a splitting

$$T(\Sigma \times \mathbb{R}) = \xi \oplus \chi$$

where χ is a 3-dimensional subbundle and ξ is a 1-dimensional subbundle of the tangent bundle. A section of the bundle ξ is the line element. Now let $\Sigma \times_{\Theta} \mathbb{R}$ be an exotic 4-manifold, i.e. $\Sigma \times_{\Theta} \mathbb{R}$ is homeomorphic to $\Sigma \times \mathbb{R}$ but not diffeomorphic to it. Usually the cross product \times will be only understand topologically. Here we extend it to the smooth situation as well. But we know that every 3-manifold Σ has a unique smoothness structure. Therefore $\Sigma \times \mathbb{R}$ represents smoothly the standard structure and we choose $\Sigma \times_{\Theta} \mathbb{R}$ to indicate the exotic smoothness structure. From this point of view we have a big difference between the bundle $T(\Sigma \times \mathbb{R})$ and $T(\Sigma \times_{\Theta} \mathbb{R})$. The first bundle admits a splitting

$$T(\Sigma \times \mathbb{R}) = T\Sigma \times T\mathbb{R}$$

whereas the second is not splittable in that manner. But it is known that the tangent bundle of every 3-manifold is trivial [9], i.e. $T\Sigma = \Sigma \times \mathbb{R}^3$. From this fact we obtain a splitting

$$T(\Sigma \times_{\Theta} \mathbb{R}) = \xi \oplus \chi$$

as needed for a Lorentz structure. This splitting is induced by the embedding $\Sigma \rightarrow \Sigma \times_{\Theta} \mathbb{R}$ and by the bundle splitting

$$T\Sigma = \Sigma \times \mathbb{R}^3 = (\Sigma \times \mathbb{R}^2) \oplus (\Sigma \times \mathbb{R}^1)$$

into a 1- and 2-dimensional subbundle. Therefore the exotic 4-manifold $\Sigma \times_{\Theta} \mathbb{R}$ admits also a Lorentz structure but of a different kind. Now one has to study a codimension-1 foliation of the 3-manifold Σ , i.e. one has the usual splitting (3-space \times time) for submanifolds (the leaves) of Σ only.

2. The Example $S^3 \times \mathbb{R}$

In this section we will study a specific example. For simplicity we choose $S^3 \times \mathbb{R}$ with a global time function so that $S^3 \times \{t\}$ is the leaf for every $t \in \mathbb{R}$ in this section. So, we obtain a foliation of $S^3 \times \mathbb{R}$. Now we discuss the change of the smoothness structure leading to a change of the foliation for $S^3 \times \mathbb{R}$. One of the first examples of an exotic smoothness structure on this manifold was given by Freedman [10] using the Poincare sphere. Lets denote the exotic $S^3 \times \mathbb{R}$ by $S^3 \times_{\Theta} \mathbb{R}$. The foliation of $S^3 \times_{\Theta} \mathbb{R}$ contains a Poincare sphere as smooth cross section (see Theorem 4 in [10]). In Appendix B, we will give the details of the construction. Here we will only present a short outline. One starts with a homology 3-sphere P , i.e. a compact 3-manifold P with the same homology as the 3-sphere but non-trivial fundamental group, see Appendix A. The Poincare sphere is one example of a homology 3-manifold. Now we consider the 4-manifold $P \times [0, 1]$ with the same fundamental group $\pi_1(P \times [0, 1]) = \pi_1(P)$. By a special procedure (the plus construction see [11, 12]), one can "kill" the fundamental group $\pi_1(P)$ in the interior of $P \times [0, 1]$. This procedure will result in a 4-manifold W with boundary $\partial W = -P \sqcup S^3$ ($-P$ with opposite orientation), a so-called cobordism between P and S^3 . The gluing $-W \cup_P W$ along P with the boundary $\partial(-W \cup_P W) = -S^3 \sqcup S^3$ defines one piece of the exotic $S^3 \times_{\Theta} \mathbb{R}$. The whole construction can be extended to both directions to get the desired exotic $S^3 \times_{\Theta} \mathbb{R}$ (see the Appendix B for details). There is one critical point in the construction: the 4-manifold W is not a smooth manifold. As Freedman [13] showed, the 4-manifold W always exists topologically but by a result of Gompf [14] (using Donaldson [15]) not smoothly. The Poincare sphere or the Brieskown sphere $\Sigma(2, 3, 7)$ are examples of homology 3-spheres P leading to a non-smooth W whereas the Brieskorn sphere $\Sigma(2, 5, 7)$ produces a smooth 4-manifold W .

The 4-manifold $-W \cup_P W$ is also non-smoothable and we will get a smoothness structure only for the non-compact $S^3 \times_{\Theta} \mathbb{R}$ (see [16]). But $S^3 \times_{\Theta} \mathbb{R}$ contains $-W \cup_P W$ with the smooth cross section P . From the physical point of view we interpret $-W \cup_P W$ as a time line of a cosmos starting as 3-sphere changing to the homology 3-sphere P and changing back to the 3-sphere. But this process is part of every exotic smoothness structure $S^3 \times_{\Theta} \mathbb{R}$, i.e. we obtain

In the exotic $S^3 \times_{\Theta} \mathbb{R}$ we have a change of the spatial topology from the 3-sphere to some homology 3-sphere.

But this conclusion is only part of the story. In cosmology one usually consider an

isotropic and homogenous model for the cosmos. Then the spatial 3-manifold has to admit a homogenous geometry with constant curvature. Obviously the 3-sphere has positive curvature as well as the Poincare 3-sphere (using Thurstons geometrization conjecture as proven by Perelman). So, from the geometrical point of view nothing changes for an exotic $S^3 \times_{\Theta} \mathbb{R}$ constructed from the Poincare sphere. But consider now the homology sphere $\Sigma(8_{10})$ with hyperbolic geometry constructed in the Appendix C. Then the corresponding $S^3 \times_{\Theta} \mathbb{R}$ contains a change of the topology and geometry from spherical 3-sphere to a hyperbolic $\Sigma(8_{10})$ and back.

In the exotic $S^3 \times_{\Theta} \mathbb{R}$ there is a change of the geometry from the spherical 3-sphere to some homology 3-sphere with geometry of postive or negative curvature.

3. Exotic cosmology and Dark energy

Our choice of the example in the previous subsection was not arbitrary. Given a compact 3-manifold Σ , the connected sum[‡] $\Sigma \# S^3$ is diffeomorphic to Σ . Therefore $\Sigma \times \mathbb{R}$ contains in some sense the example $S^3 \times \mathbb{R}$. Especially we can generalize the conclusions above to this case as well. Now we will study the geometry and topology changing process more carefully. Lets consider the Robertson-Walker metric (with $c = 1$)

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k$$

with the scaling function $a(t)$. At first we assume a spacetime $S^3 \times \mathbb{R}$ with increasing function $a(t)$ fulfilling the Friedman equations

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{k}{a(t)^2} = \kappa \frac{\rho}{3} \quad (1)$$

$$2 \left(\frac{\ddot{a}(t)}{a(t)}\right) + \left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{k}{a(t)^2} = -\kappa p \quad (2)$$

derived from Einsteins equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad (3)$$

with the gravitational constant κ and the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu} \quad (4)$$

with the (time-dependent) energy density ρ and the (time-dependent) pressure p . The spatial cosmos has the scalar curvature 3R

$${}^3R = \frac{k}{a^2}$$

from the metric h_{ik} and we obtain the 4-dimensional scalar curvature R

$$R = \frac{6}{a^2} (\ddot{a} \cdot a + \dot{a}^2 + k) \quad (5)$$

or

$$R = 3\kappa(\rho - 3p)$$

[‡] Let M, N be a compact n -manifolds. The connected sum $M \# N$ is the procedure to cut out a disk D^n from the interior $\text{int}(M) \setminus D^n$ and $\text{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \sqcup \partial M$ and $S^{n-1} \sqcup \partial N$, respectively, and glue them together (by a smooth map) along the common boundary component S^{n-1} . The boundary $\partial(M \# N) = \partial M \sqcup \partial N$ is the disjoint union of the boundaries $\partial M, \partial N$.

with the acceleration

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) .$$

Let us consider the model $S^3 \times \mathbb{R}$ with positive spatial curvature $k = +1$. In case of dust matter ($p = 0$) only, one obtains a closed universe. But for negative spatial curvature $k = -1$, one has an open universe. Now we consider our model of an exotic $S^3 \times_{\Theta} \mathbb{R}$, see section 2. As explained above, the foliation of $S^3 \times_{\Theta} \mathbb{R}$ must contain a homology 3-sphere $\Sigma(8_{10})$ with negative curvature. But then we have a transition from a space with positive curvature to a space with negative curvature and back. In the Appendix B, this period was denoted by $-W_1 \cup W_1$ where both W_1 were glued along $\Sigma(8_{10})$. At first, in all "later" versions of the spatial space we have the space $\Sigma(8_{10})$. Secondly, hyperbolic 3-manifolds like $\Sigma(8_{10})$ have a special property, called Mostow rigidity [17]. It means, that a hyperbolic 3-manifold can not be scaled, i.e. the volume or the curvature is a topological invariant. Therefore, if the hyperbolic 3-manifold Σ is part of the space then this part has a constant curvature. We will use this property in the following.

Let us assume that the period of the transition starts at time t_0 with negligible length. So, for a time $t > t_0$ we have the effective equation

$$2 \left(\frac{\ddot{a}(t)}{a(t)} \right) + \left(\frac{\dot{a}(t)}{a(t)} \right)^2 + \frac{1}{a(t)^2} - \frac{1}{(a(t_0))^2} = -\kappa p$$

with the constant $\Lambda = (a(t_0))^{-2}$ proportional to the constant curvature of Σ . The first equation (1) is modified in the same manner. The volume of $\Sigma(8_{10})$ is proportional to $(a(t_0))^3$. Finally we obtain the acceleration

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{2} - \frac{\kappa}{6}(\rho + 3p) .$$

This acceleration can be positive, i.e. for a time $t \gg t_0$ the term Λ dominates and we obtain an accelerated expansion.

4. Conclusion

In this paper we discuss spacetime models of the universe with an exotic smoothness structure. In four dimensions there is an infinity of possible exotic structures. So, it will be rather a surprise if our universe admits the standard smoothness structure. Here we paid special attention to the model $S^3 \times \mathbb{R}$ and modify it to the exotic $S^3 \times_{\Theta} \mathbb{R}$. Then we obtain a topological transition from the space S^3 to a hyperbolic homology 3-sphere $\Sigma(8_{10})$ (see Appendix C). This transition can be interpreted as dark energy which forces the expansion to accelerate. Therefore exotic smoothness can serve as another way to obtain the accelerated expansion. Of course the value of the cosmological constant Λ is not determined yet. But this value must be much smaller than every estimate of quantum field theory. In our forthcoming work we will try explain the value.

Appendix A. Homology 3-spheres

A homology 3-sphere Σ is a compact, connected 3-manifold without boundary having the same homology as the 3-sphere S^3 . This definition implies the homology groups of Σ :

$$H_0(\Sigma) = H_3(\Sigma) = \mathbb{Z}, \quad H_1(\Sigma) = H_2(\Sigma) = 0$$

This definition seem to restrict the 3-manifolds very strong but there is one invariant characterizing 3-manifolds rather uniquely: the fundamental group $\pi_1(\Sigma)$ (closed curves up to homotopy). The first example of a homology 3-sphere was constructed by Poincaré using the binary icosader group $I^* = \langle s, t \mid s^5 = (st)^2, t^3 = (st)^2 \rangle$, i.e. the group of sequences generated by s, t and its inverses s^{-1}, t^{-1} restricted by the relations $s^5 = (st)^2, t^3 = (st)^2$. The group action $S^3 \times I^* \rightarrow S^3$ is free and one has the equivalence classes S^3/I^* forming a smooth manifold, the *Poincaré sphere*. The fundamental group is given by $\pi_1(S^3/I^*) = I^*$. The group I^* has a very important property: I^* is perfect. For tht purpose we define the commutator $[s, t] = sts^{-1}t^{-1}$ in a group G and denote with $[G, G]$ the subgroup generated by the commutators of G . One calls a group *perfect* iff $G = [G, G]$. Now we need a relation between the first homology group and the fundamental group given by

$$H_1(M) = \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]}$$

Then the vanishing of $H_1(M)$ is equivalent to a perfect fundamental group $\pi_1(M) = [\pi_1(M), \pi_1(M)]$. Therefore we obtain from the compactness $H_0(S^3/I^*) = H_3(S^3/I^*) = \mathbb{Z}$ and $H_1(S^3/I^*) = H_2(S^3/I^*) = 0$ from $I^* = [I^*, I^*]$ and duality. The group I^* has 120 elements and is the *only finite, perfect* group.

Appendix B. Constructing exotic $S^3 \times \mathbb{R}$'s

Given a homology 3-sphere Σ which do not bound a contractable, smooth 4-manifold. According to Freedman [13], every homology 3-sphere bounds a contractable, topological 4-manifold but not every of these 4-manifolds is smoothable. Now we consider the following pieces: W_1 as cobordism between Σ and its one-point complement $\Sigma \setminus pt.$ as well W_2 as cobordism between $\Sigma \setminus pt.$ and $\Sigma \setminus pt.$. The manifold $W = \dots \cup -W_2 \cup -W_2 \cup -W_1 \cup W_1 \cup W_2 \cup W_2 \cup \dots$ (see [10]) is homeomorphic to $S^3 \times \mathbb{R}$ (using the proper h-cobordism theorem in [13]) but not diffeomorphic to it, i.e. $W = S^3 \times_{\Theta} \mathbb{R}$. The construction of the pieces W_1, W_2 rely heavily on the concept of a Casson handle. For that purpose we consider a Casson handle CH and its 3-stage tower T_3^1 . By the embedding theorem of Freedman (see [10], Theorem 1), one can construct another 3-stage tower T_3^1 inside of T_3^0 (increase the number of self-intersections of the core). This process can be done infinitely. Lets take an example of an homology 3-sphere Σ constructed in the next section. The fundamental group is generated by one generator, i.e. we need a single Casson handle only to kill this generator by defining an embedding $T_3^0 \hookrightarrow \Sigma \times [0, 1]$. Now by killing an arc in each 3-stage tower $T_{3,arc}^0 = T_3^0 \setminus \{\text{arc}\}$ and by killing a line $T_{3,line}^0 = T_3^0 \setminus \{\text{line}\}$ we can form the desired cobordisms W_1, W_2 above: $W_1 = \Sigma \times [0, 1] \setminus \bigcap_{i=0}^{\infty} T_{3,arc}^i$ and $W_2 = \Sigma \times [0, 1] \setminus \bigcap_{i=0}^{\infty} T_{3,line}^i$ completing the construction of the exotic $S^3 \times \mathbb{R}$. Furthermore we obtain the smooth cross section Σ in the part $-W_1 \cup W_1$ of W .

Appendix C. Constructing a hyperbolic homology 3-sphere

Here we will construct one example of a hyperbolic homology 3-sphere which does not bound a contractable 4-manifold. Then we can use the procedure above to get an exotic $S^3 \times_{\Theta} \mathbb{R}$. For that purpose we consider a knot K , i.e. a smooth embedding $S^1 \rightarrow S^3$. This knot K can be thickened to $N(K) = K \times D^2$ and one obtains the knot complement $C(K) = S^3 \setminus N(K)$ with boundary $\partial C(K) = T^2$. By the attachment of a solid torus

$D^2 \times S^1$ using a -1 Dehn twist, one obtains the manifold $\Sigma = C(K) \cup (D^2 \times S^1)$, a homology 3-sphere [18]. The geometry of Σ is determined by the knot complement $C(K)$. A fundamental result of Thurston [19] states that most knot complements are hyperbolic 3-manifolds. Examples are the figure-8 knot 4_1 , the 3-twist knot 5_2 or the knot 8_{10} (in Rolfsen notation [18]). Therefore we have to look for a knot inducing a hyperbolic knot complement and leading to a homology 3-sphere which does not bound a smooth contractible 4-manifold. As Freedman [13] showed, every homology 3-sphere bounds a contractible, topological 4-manifold. But Donaldson [15] found the first example, the Poincaré sphere, of a homology 3-sphere which fails to do it. The Poincaré sphere is generated by the trefoil knot 3_1 using the procedure above. Every homology 3-sphere homology-cobordant to the Poincaré sphere has the same property [20]. From the knot-theoretical point of view, we have to look for a knot concordant to the trefoil knot [21]. One example is given by the knot 8_{10} (see [22]). Then the homology 3-sphere Σ constructed from this knot has all desired properties.

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